2023

AP

# AP® Calculus BC

## Sample Student Responses and Scoring Commentary

### Inside:

Free-Response Question 1

- ☑ Scoring Guidelines
- ☑ Student Samples
- ☑ Scoring Commentary

## Part A (AB or BC): Graphing calculator required

## Question 1 — 9 points

**General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

| $t$ (seconds) | 0 | 60 | 90 | 120 | 135 | 150 |
|---|---|---|---|---|---|---|
| $f(t)$ (gallons per second) | 0 | 0.1 | 0.15 | 0.1 | 0.05 | 0 |

A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function $f$, where $f(t)$ is measured in gallons per second and $t$ is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

| | Model Solution | Scoring | |
|---|---|---|---|
| **(a)** | Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t)\,dt$. | | |
| | $\int_{60}^{135} f(t)\,dt$ represents the total number of gallons of gasoline pumped into the gas tank from time $t = 60$ seconds to time $t = 135$ seconds. | Interpretation with units | **1 point** |
| | $\int_{60}^{135} f(t)\,dt$ $\approx f(90)(90-60) + f(120)(120-90) + f(135)(135-120)$ | Form of Riemann sum | **1 point** |
| | $= (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25$ | Answer | **1 point** |

**Scoring notes:**

- To earn the first point the response must reference gallons of gasoline added/pumped and the time interval $t = 60$ to $t = 135$.
- To earn the second point at least five of the six factors in the Riemann sum must be correct.
- If there is any error in the Riemann sum, the response does not earn the third point.
- A response of $(0.15)(30) + (0.1)(30) + (0.05)(15)$ earns both the second and third points, unless there is a subsequent error in simplification, in which case the response would earn only the second point.

- A response that presents a correct value with accompanying work that shows the three products in the Riemann sum but does not show all six of the factors and/or the sum process, does not earn the second point but does earn the third point. For example, responses of either $4.5 + 3.0 + 0.75$ or $(0.15)(30), 0.1(30), 0.05(15) \rightarrow 8.25$ earn the third point but not the second.
- A response of $f(90)(90 - 60) + f(120)(120 - 90) + f(135)(135 - 120) = 8.25$ earns both the second and the third points.
- A response that presents an answer of only $8.25$ does not earn either the second or third point.
- A response that provides a completely correct left Riemann sum with accompanying work, $f(60)(30) + f(90)(30) + f(120)(15) = 9,$ or $(0.1)(30) + (0.15)(30) + (0.1)(15)$ earns 1 of the last 2 points. A response with any errors or missing factors in a left Riemann sum earns neither of the last 2 points.

**Total for part (a) 3 points**

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**(b)** Must there exist a value of $c$, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

| | | |
|---|---|---|
| $f$ is differentiable. $\Rightarrow$ $f$ is continuous on $[60, 120]$.<br><br>$\dfrac{f(120) - f(60)}{120 - 60} = \dfrac{0.1 - 0.1}{60} = 0$<br><br>By the Mean Value Theorem, there must exist a $c$, for $60 < c < 120$, such that $f'(c) = 0$. | $f(120) - f(60) = 0$ | **1 point** |
| | Answer with justification | **1 point** |

**Scoring notes:**

- To earn the first point a response must present either $f(120) - f(60) = 0$, $0.1 - 0.1 = 0$ (perhaps as the numerator of a quotient), or $f(60) = f(120)$.
- To earn the second point a response must:
  - o have earned the first point,
  - o state that $f$ is continuous because $f$ is differentiable (or equivalent), and
  - o answer "yes" in some way.
- A response may reference either the Mean Value Theorem or Rolle's Theorem.
- A response that references the Intermediate Value Theorem cannot earn the second point.

**Total for part (b) 2 points**

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**(c)** The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$. Show the setup for your calculations.

| | | |
|---|---|---|
| $\dfrac{1}{150 - 0}\displaystyle\int_0^{150} g(t)\, dt$ | Average value formula | **1 point** |
| $= 0.0959967$ | Answer | **1 point** |

| | |
|---|---|
| The average rate of flow of gasoline, in gallons per second, is $0.096$ (or $0.095$). | |

**Scoring notes:**

- The exact value of $\frac{1}{150}\int_0^{150} g(t)\,dt$ is $\frac{12}{125}\sin\left(\frac{25}{16}\right)$.
- A response may present the average value formula in single or multiple steps. For example, the following response earns both points: $\int_0^{150} g(t)\,dt = 14.399504$ so the average rate is $0.0959967$.
- A response that presents the average value formula in multiple steps but provides incorrect or incomplete communication (e.g., $\int_0^{150} g(t)\,dt = \frac{14.399504}{150} = 0.0959967$) earns 1 out of 2 points.
- A response of $\int_0^{150} g(t)\,dt = 0.0959967$ does not earn either point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{150}\int_0^{150} g(t)\,dt = 0.149981$ or $0.002618$.

**Total for part (c) 2 points**

**(d)** Using the model $g$ defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

| | | |
|---|---|---|
| $g'(140) \approx -0.004908$<br>$g'(140) = -0.005$ (or $-0.004$) | $g'(140)$ | **1 point** |
| The rate at which gasoline is flowing into the tank is decreasing at a rate of $0.005$ (or $0.004$) gallon per second per second at time $t = 140$ seconds. | Interpretation | **1 point** |

**Scoring notes:**

- The exact value of $g'(140)$ is $\frac{1}{500}\cos\left(\frac{49}{36}\right) - \frac{49}{9000}\sin\left(\frac{49}{36}\right)$.
- The value of $g'(140)$ may appear only in the interpretation.
- To be eligible for the second point a response must present some numerical value for $g'(140)$.
- To earn the second point the interpretation must include "the rate of flow of gasoline is changing at a rate of [the declared value of $g'(140)$]" and "at $t = 140$" (or equivalent).
- An interpretation of "decreasing at a rate of $-0.005$" or "increasing at a rate of $0.005$" does not earn the second point.
- Degree mode: In degree mode, $g'(140) = 0.001997$ or $0.00187$.

**Total for part (d) 2 points**

**Total for question 1 9 points**

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**Answer QUESTION 1 parts (a) and (b) on this page.**

| $t$ (seconds) | 0 | 60 | 90 | 120 | 135 | 150 |
|---|---|---|---|---|---|---|
| $f(t)$ (gallons per second) | 0 | 0.1 | 0.15 | 0.1 | 0.05 | 0 |

**Response for question 1(a)**

$(90-60)(.15) + (120-90)(.1) + (135-120)(.05) = 8.25$ gallons

$\int_{60}^{135} f(t)\,dt$ represents the amount of gas pumped, in gallons, from $t=60$ to $t=135$ seconds.

**Response for question 1(b)**

$f(t)$ is always differentiable, and therefore it must be continuous on $[a, b]$. $a=60$ & $b=120$.

According to Rolle's Theorem, if $f(a) = f(b)$ which is true because $f(60) = .1 = f(120)$, there must be some $x$ value, "c", at which $f'(x) = 0$.

Therefore, there must be a value of $c$ such that $f'(c) = 0$

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

**Answer QUESTION 1 parts (c) and (d) on this page.**

Response for question 1(c)

avg $= \frac{1}{150-0}\int_0^{150} g(t)\,dt = \boxed{.096 \text{ gallons/second}}$

Response for question 1(d)

$g'(140) = -0.005$ gallons/second/second

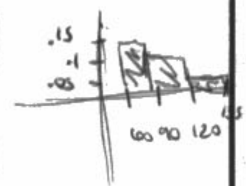
math 8

$g'(140) = -.005$ means that at $t = 140$ seconds, the rate of flow of gasoline is changing at a rate of $-.005$.

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

**Answer QUESTION 1 parts (a) and (b) on this page.**

| $t$ (seconds) | 0 | 60 | 90 | 120 | 135 | 150 |
|---|---|---|---|---|---|---|
| $f(t)$ (gallons per second) | 0 | 0.1 | 0.15 | 0.1 | 0.05 | 0 |




**Response for question 1(a)**

$\int_{60}^{135} f(t)\,dt$ in the context of the problem means/establishes the time frame (from 60 to 135 seconds) in which ~~a~~ a certain amount of gallons of gasoline are pumped into a gas tank.

$(30 \cdot .15) + (30 \cdot .1) + (30 \cdot .05)$

$4.5 + 3 + 1.5 = \boxed{9 \text{ gallons}}$

**Response for question 1(b)**

Yes, there must exist a value of c for $60<c<120$ such that $f'(c)=0$ because in that time, the $f(t)$ fluctuates from .1 gal/s, to .15 gal/sec, and back to .1 gal/s. Given this, at at least 1 instantaneous point in that timeframe, $f'(c)$ must $= 0$ as the rate increases and then decreases.

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

**Answer QUESTION 1 parts (c) and (d) on this page.**

**Response for question 1(c)**

$$\frac{1}{b-a}\int_a^b f(t)\,dt$$

$$\frac{1}{150-0}\int_0^{150} \frac{t}{500}\cos\left(\left(\frac{t}{120}\right)^2\right)dt$$

$$\frac{1}{150}\int_0^{150} \frac{t}{500}\cos\left(\left(\frac{t}{120}\right)^2\right)dt$$

$$= \boxed{0.0\text{9}\ldots \text{ gal/sec}}$$

**Response for question 1(d)**

$g'(140) = ?$ $\quad g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$

$$\frac{d}{dt}g(t) = \frac{d}{dt}\left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)_{t=140}$$

$$= \boxed{-0.00491 \text{ gal/s}^2}$$

This figure represents the acceleration at which the gasoline's velocity into the gas tank is functioning - so as ~~time~~ gas is pumped, the velocity at which it is pumped decelerates by $-0.00491$ gal/s$^2$.



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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**Sample 1C**

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**Answer QUESTION 1 parts (a) and (b) on this page.**

| $t$ (seconds) | 0 | 60 | 90 | 120 | 135 | 150 |
|---|---|---|---|---|---|---|
| $f(t)$ (gallons per second) | 0 | 0.1 | 0.15 | 0.1 | 0.05 | 0 |

**Response for question 1(a)**

$\int_{60}^{135} f(t)\,dt \rightarrow 15(0.05)+30(0.1)+30(0.15) = 8.25$ gls

**Response for question 1(b)**

Yes because c lies in the integral $60<c<120$ on the graph

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**Answer QUESTION 1 parts (c) and (d) on this page.**

Response for question 1(c)

$g(t) = \left[\frac{t}{500}\right] \cos\left(\left(\frac{t}{120}\right)^2\right)$

$0 \leq t \leq 150$

$\int_0^{150} \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)dt = 14.399$ g/s

Response for question 1(d)

$\int_0^{150} \left(\frac{140}{500}\right)\cos\left(\left(\frac{140}{120}\right)^2\right)dt = 8.742$

# Question 1

**Note:** Student samples are quoted verbatim and may contain spelling and grammatical errors.

## Overview

In this problem students were given a table of times $t$ in seconds and values of a function $f(t)$, which models the rate of flow of gallons of gasoline pumped into a gas tank.

In part (a) students were asked to interpret the meaning of $\int_{60}^{135} f(t)\, dt$ using correct units. Then students were asked to use a right Riemann sum with three subintervals to approximate the value of this integral. A correct response will indicate that the integral represents the accumulated gallons of gasoline pumped into the tank during the time interval from $t = 60$ to time $t = 135$ seconds. The approximation is found using the following expression: $(90 - 60) \cdot f(90) + (120 - 90) \cdot f(120) + (135 - 120) \cdot f(135)$.

In part (b) students were asked to justify whether there must be a value of $c$, with $60 < c < 120$, such that $f'(c) = 0$. Students are expected to note that because the function $f$ is known to be differentiable on the interval $(0, 150)$, it must be continuous on the subinterval $[60, 120]$. Therefore, because the average rate of change of $f$ on the interval $[60, 120]$ is equal to $0$, such a value of $c$ is guaranteed by the Mean Value Theorem.

In part (c) the function $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ was introduced as a second function that modeled the rate of flow of the gasoline. Students were asked to use the model $g$ to find the average rate of flow of the gasoline over the time interval $0 \le t \le 150$. A correct response will show the setup $\frac{1}{150 - 0} \cdot \int_0^{150} g(t)\, dt$ and then use a calculator to find the value $0.096$ gallon per second.

In part (d) students were asked to find the value of $g'(140)$ and interpret the meaning of this value in the context of the problem. A correct response will use a calculator to find $g'(140) = -0.005$ and report that at time $t = 140$ seconds the rate at which gasoline is flowing into the tank is decreasing at a rate of $0.005$ gallon per second per second.

**Sample: 1A**
**Score: 9**

The response earned 9 points: 3 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d).

In part (a) the response earned the first point with the statement "the amount of gas pumped, in gallons, from $t = 60$ to $t = 135$ seconds." The response earned the second point for the correct form of the Riemann sum. The response earned the third point for the correct answer.

In part (b) the response earned the first point for "$f(60) = .1 = f(120)$." The response earned the second point because it earned the first point, states that "$f(t)$ is always differentiable, and therefore it must be continuous on $[a, b]$ $a = 60$ & $b = 120$," and states the correct conclusion.

In part (c) the response earned the first point with the inclusion of the average value formula. The response earned the second point with the correct answer.

## Question 1 (continued)

In part (d) the response earned the first point for the correct value of $g'(140)$. The response earned the second point with the statement "at $t = 140$ seconds, the rate of flow of gasoline is changing at a rate of $-.005$."

**Sample: 1B**
**Score: 5**

The response earned 5 points: 2 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the first point with the statement "the time frame (from $60$ to $135$ seconds) in which a certain amount of gallons of gasoline are pumped into a gas tank." The response earned the second point for the correct form of the Riemann sum with five of the six factors correct. The third point was not earned because the response contains an error in the Riemann sum.

In part (b) the response did not earn the first point because the expression $f(120) - f(60) = 0$ is not included. Because the first point was not earned, the response is not eligible for the second point. In addition, the response did not earn the second point because the response does not state that $f$ is continuous because $f$ is differentiable.

In part (c) the response earned the first point because the response includes the average value formula. The response earned the second point with the correct answer.

In part (d) the response earned the first point with the presence of the correct value of $g'(140)$. The response did not earn the second point because the response does not interpret the declared value of $g'(140)$ correctly (it needs to discuss a rate of a rate). The words acceleration and velocity should be used to refer to an object in motion.

**Sample: 1C**
**Score: 2**

The response earned 2 points: 2 points in part (a), no points in part (b), no points in part (c), and no points in part (d).

In part (a) the response did not earn the first point. The response earned the second point for the correct form of the Riemann sum. The response earned the third point for the correct answer.

In part (b) the response did not earn the first point because the response does not include $f(120) - f(60) = 0$. Because the first point was not earned, the response is not eligible for the second point. In addition, the response did not earn the second point because the response does not state that $f$ is continuous because $f$ is differentiable.

In part (c) the response did not earn the first point because the response does not include the average value formula. The response did not earn the second point because the response does not include the correct answer.

In part (d) the response did not earn the first point because the response does not include the value of $g'(140)$. The response did not earn the second point because the response does not include the correct interpretation.

2023

# AP® Calculus BC

## Sample Student Responses and Scoring Commentary

**Inside:**

Free-Response Question 2

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

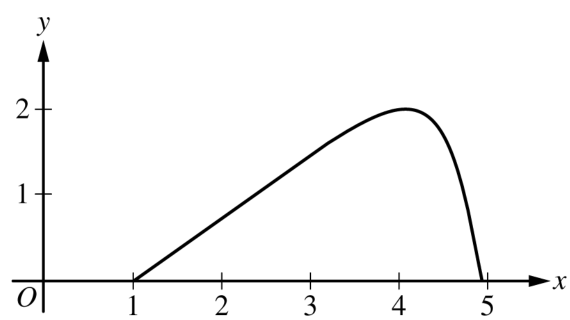
## Part A (BC): Graphing calculator required

## Question 2 — 9 points

**General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.




For $0 \le t \le \pi$, a particle is moving along the curve shown so that its position at time $t$ is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

| Model Solution | Scoring | |
|---|---|---|
| **(a)** Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations. | | |
| $x''(1) = \frac{d}{dt}\left(e^{\cos t}\right)\Big\|_{t=1} = -1.444407$ | $x''(1)$ with setup | **1 point** |
| $y(t) = 2\sin t \Rightarrow y'(t) = 2\cos t$<br>$y''(1) = \frac{d}{dt}(2\cos t)\Big\|_{t=1} = -1.682942$<br>The acceleration vector at time $t = 1$ is $a(1) = \langle -1.444, -1.683 \text{ (or } -1.682)\rangle$. | $y''(1)$ with setup | **1 point** |

**Scoring notes:**

- The exact answer is $\langle x''(1), y''(1)\rangle = \left\langle -e^{\cos 1}\sin 1, -2\sin 1\right\rangle$.
- $\left\langle -e^{\cos t}\sin t, -2\sin t\right\rangle$ together with an incorrect or missing evaluation at $t = 1$ earns 1 of the 2 points.

- A response of $\left\langle -e^{\cos t}\sin t, -2\sin t\right\rangle = \left\langle -e^{\cos 1}\sin 1, -2\sin 1\right\rangle$ or equivalent earns only 1 of the 2 points because it equates an expression to a numerical value.
- An unsupported correct acceleration vector earns 1 of the 2 points.
- The acceleration vector may be presented with other symbols, for example $( \,,\, )$ or $[\,,\,]$.
- The components may be listed separately, as long as they are labeled.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $x''(1) = -0.000828$ or $-0.047433$ and $y''(1) = -0.000609$ or $-0.034905$. A response that presents one of these values with correct setups earns 1 of the 2 points.

**Total for part (a) 2 points**

**(b)** For $0 \le t \le \pi$, find the first time $t$ at which the speed of the particle is $1.5$. Show the work that leads to your answer.

| | | |
|---|---|---|
| Speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2}$<br><br>$0 \le t \le \pi$ and $\sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2} = 1.5$ | $\sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2} = 1.5$ | **1 point** |
| $\Rightarrow t = 1.254472,\ t = 2.358077$<br><br>The first time at which the speed of the particle is $1.5$ is $t = 1.254$. | Answer | **1 point** |

**Scoring notes:**

- A response with an implied equation is eligible for both points. For example, a response of "Speed $= \sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2}$ and is first equal to $1.5$ at $t = 1.254$" earns both points.
- $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1.5$ earns the first point. Speed $= 1.5$ by itself does not earn the first point. Both of these responses are eligible to earn the second point.
- A response need not consider the value $t = 2.358077$.
- A response of $t = 1.254$ alone does not earn either point.
- A response with a parenthesis error(s) in either $\left(e^{\cos t}\right)^2$ or $(2\cos t)^2$ does not earn the first point but does earn the second point for the correct answer. Note: $\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$ is not considered a parenthesis error.

- Degree mode: In degree mode, $\sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2} = 1.5$ has no solution for $0 \le t \le \pi$.

  A response that finds no time $t$ at which the speed of the particle is $1.5$ cannot be assumed to be working in degree mode.

**Total for part (b) 2 points**

**(c)** Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the $x$-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

| | | |
|---|---|---|
| $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{e^{\cos t}}$<br>$\left.\frac{dy}{dx}\right\vert_{t=1} = \frac{2\cos 1}{e^{\cos 1}} = 0.629530$<br>The slope of the line tangent to the curve at $t = 1$ is $0.630$ (or $0.629$). | Slope with supporting work | **1 point** |
| $x(1) = x(0) + \int_0^1 \frac{dx}{dt}\,dt = 1 + \int_0^1 e^{\cos t}\,dt = 3.341575$ | $\int_0^1 e^{\cos t}\,dt$ | **1 point** |
| The $x$-coordinate of the position at $t = 1$ is $3.342$ (or $3.341$). | $x(1)$ | **1 point** |

**Scoring notes:**

- To earn the first point, the response must communicate $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; for example:
  - $\frac{dy}{dx} = \frac{2\cos 1}{e^{\cos 1}}$
  - $\frac{dy/dt}{dx/dt} = 0.63$
  - $x'(1) = 1.716526, \; y'(1) = 1.080605, \; \text{slope} = 0.63$
  - $\frac{dy}{dt} = 2\cos t, \; \text{slope} = 0.63$
- A response may import an incorrect expression for $y'(t)$ or value of $y'(1)$ from part (a), provided it was declared in part (a).
- The second point is earned for a response that presents the definite integral $\int_0^1 e^{\cos t}\,dt$ or $\int_0^1 \frac{dx}{dt}\,dt$ with or without the initial condition.

- For the second point, if the differential is missing:
  - $\int_0^1 e^{\cos t}$ earns the second point and is eligible for the third point.
  - $x(1) = \int_0^1 e^{\cos t}$ earns the second point but is not eligible for the third point.
  - $x(1) = 1 + \int_0^1 e^{\cos t}$ earns the second point and is eligible for the third point.
  - $x(1) = \int_0^1 e^{\cos t} + 1$ does not earn the second point but earns the third point for the correct answer.
- The third point is not earned for a response that presents an incorrect statement, such as $x(1) = \int_0^1 e^{\cos t}\, dt = 1 + 2.342.$
- Degree mode: In degree mode, $\frac{dy}{dx} = 0.735759$ or $0.012841$ and $1 + \int_0^1 e^{\cos t}\, dt = 3.718144.$

**Total for part (c) 3 points**

**(d)** Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

| | | |
|---|---|---|
| $\int_0^{\pi} \sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2}\, dt$ | Integral | **1 point** |
| $= 6.034611$<br><br>The total distance traveled by the particle over $0 \le t \le \pi$ is $6.035$ (or $6.034$). | Answer | **1 point** |

**Scoring notes:**

- The first point is earned for presenting the correct integrand in a definite integral.
- Parentheses errors were assessed in part (b) and, therefore, will not affect the scoring in part (d).
- If the integrand is an incorrect speed function imported from part (b), the response earns the first point and does not earn the second point.
- An unsupported answer of $6.035$ (or $6.034$) does not earn either point.
- Degree mode: In degree mode, the total distance is $10.596835$ or $8.536161$.

**Total for part (d) 2 points**

**Total for question 2 9 points**

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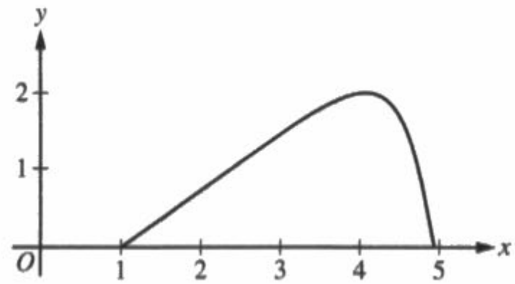
**Answer QUESTION 2 parts (a) and (b) on this page.**




**Response for question 2(a)**

$y(t) = 2\sin t$

$y'(t) = 2\cos t$

$v(t) = \langle e^{\cos t}, y'(t) \rangle$

$v(t) = \langle e^{\cos t}, 2\cos t \rangle$

$a(1) = \langle \frac{d}{dt}\Big|_{t=1}[e^{\cos t}], \frac{d}{dx}\Big|_{t=1}[2\cos t] \rangle$

$\boxed{a(1) = \langle -1.444, -1.683 \rangle}$

**Response for question 2(b)**

$|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$

$1.5 = \sqrt{(e^{\cos t})^2 + (2\cos t)^2}$

$\boxed{t = 1.254}$

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

**Answer QUESTION 2 parts (c) and (d) on this page.**

Response for question 2(c)

$\frac{dy}{dx} = \frac{2\cos t}{e^{\cos t}}$ $\quad \frac{dy}{dx}\Big|_{t=1} \left[\frac{2\cos t}{e^{\cos t}}\right] = \boxed{-0.451}$

$\int_0^1 e^{\cos t}\, dt = x(1) - x(0)$

$2.342 = x(1) - 1$

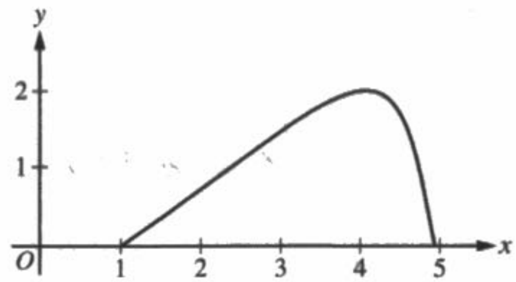
$\boxed{x(1) = 3.342}$

Response for question 2(d)

$\text{distance} = \int_0^{\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$= \boxed{6.035}$

**Answer QUESTION 2 parts (a) and (b) on this page.**



**Response for question 2(a)**

$\frac{dy}{dt} = 2\cos t$, $\frac{d^2y}{dt^2} = -2\sin t$

$\frac{dx}{dt} = e^{\cos t}$ $\frac{d^2x}{dt^2} = e^{\cos t} \cdot -\sin t$

$\frac{d^2y}{dx^2} = \langle -e^{\cos t}\sin t, -2\sin t \rangle$ at $t = 1$: $\langle -e^{\cos(1)}\sin(1), -2\sin(1) \rangle$

$\langle -1.444, -1.683 \rangle$

**Response for question 2(b)**

$\sqrt{(2(\cos t))^2 + (e^{\cos t})^2} = 1.5$

$t = 1.254$

**Answer QUESTION 2 parts (c) and (d) on this page.**

Response for question 2(c)

$\frac{dy}{dt} = 2\cos t$

$\frac{dx}{dt} = e^{\cos t}$

$\frac{dy}{dx} = \frac{2\cos t}{e^{\cos t}}$

$\left.\frac{dy}{dx}\right|_{t=0} = 0.7357$

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{2\cos(1)}{e^{\cos(1)}} = 0.629$

$y - 0 = 0.7357(x-1)$

$y = 0.7357x - 0.7357$

$y(1) = 0.7357x - 0.7357$

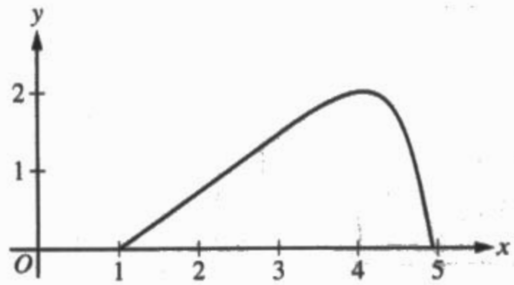
$1.6829 = 0.7357x - 0.7357 \Rightarrow x = 3.287$

Response for question 2(d)

$\frac{dy}{dx} = \frac{2\cos t}{e^{\cos t}}$

$\int_0^{\pi} \frac{2\cos t}{e^{\cos t}}\, dt = -3.551$

**Answer QUESTION 2 parts (a) and (b) on this page.**




Response for question 2(a)

$(x(t), y(t)) = (?, 2\sin t)$

$(x'(t), y'(t)) = (e^{\cos t}, 2\cos t)$

$(x''(t), y''(t) = (\cos t\, e^{\sin t}, 2 - \sin t)$

$\cancel{1}$ $\cancel{2}$

$t = 1$ ; $(.2329, -1.6829)$

Response for question 2(b)

$a_t = 1.5$

$2\sin t = 1.5$

$\sin t = .75$

$t = .85$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

~~1.6~~

$e^{\cos t}$ $\quad 2\cos t$

$$\frac{1.0806}{1.7165} = \frac{y}{x} = m$$

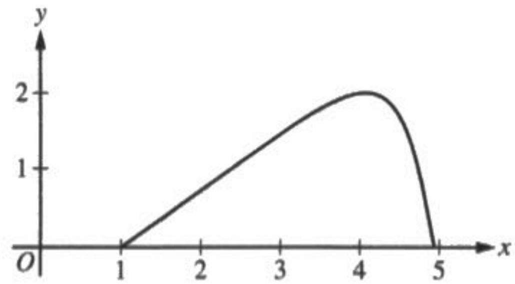
$\boxed{.62953}$

Response for question 2(d)

$x'(t) = e^{\cos(t)}$

$x(t) = ?$

**Answer QUESTION 2 parts (a) and (b) on this page.**




Response for question 2(a)

$\cancel{y'(t) = 2\cos t + c}$ $y(t) = 2\sin t$

$y'(t) = 2\cos t$

$v(t) = \langle e^{\cos t}, y'(t) \rangle$

$v(t) = \langle e^{\cos t}, 2\cos t \rangle$

$a(1) = \left\langle \frac{d}{dt}\Big|_{t=1}[e^{\cos t}], \frac{d}{dx}\Big|_{t=1}[2\cos t] \right\rangle$

$\boxed{a(1) = \langle -1.444, -1.683 \rangle}$

Response for question 2(b)

$|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$

$1.5 = \sqrt{(e^{\cos t})^2 + (2\cos t)^2}$

$\boxed{t = 1.254}$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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**Answer QUESTION 2 parts (c) and (d) on this page.**

Response for question 2(c)

$\frac{dy}{dx} = \frac{2\cos t}{e^{\cos t}}$ $\quad \frac{dy}{dx}\Big|_{t=1} \left[\frac{2\cos t}{e^{\cos t}}\right] = \boxed{-0.451}$

$\int_0^1 e^{\cos t}\, dt = x(1) - x(0)$

$2.342 = x(1) - 1$

$\boxed{x(1) = 3.342}$

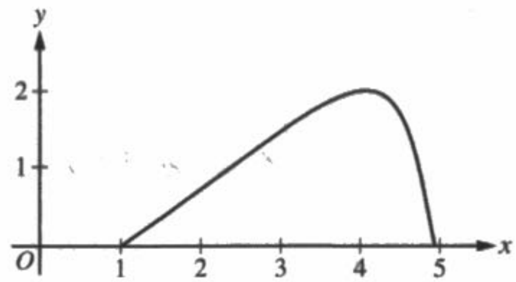
Response for question 2(d)

$\text{distance} = \int_0^{\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$= \boxed{6.035}$

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

**Answer QUESTION 2 parts (a) and (b) on this page.**



**Response for question 2(a)**

$\frac{dy}{dt} = 2\cos t$, $\frac{d^2y}{dt^2} = -2\sin t$

$\frac{dx}{dt} = e^{\cos t}$ $\frac{d^2x}{dt^2} = e^{\cos t} \cdot -\sin t$

$\frac{d^2y}{dx} = \langle -e^{\cos t}\sin t, -2\sin t \rangle$ at $t = 1$: $\langle -e^{\cos(1)}\sin(1), -2\sin(1) \rangle$

$\langle -1.444, -1.683 \rangle$

**Response for question 2(b)**

$\sqrt{(2(\cos t))^2 + (e^{\cos t})^2} = 1.5$

$t = 1.254$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5217/6

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

**Answer QUESTION 2 parts (c) and (d) on this page.**

Response for question 2(c)

$\frac{dy}{dt} = 2\cos t$

$\frac{dx}{dt} = e^{\cos t}$

$\frac{dy}{dx} = \frac{2\cos t}{e^{\cos t}}$

$\left.\frac{dy}{dx}\right|_{t=0} = 0.7357$

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{2\cos(1)}{e^{\cos(1)}} = 0.629$

$y - 0 = 0.7357(x-1)$

$y = 0.7357x - 0.7357$

$y(1) = 0.7357x - 0.7357$

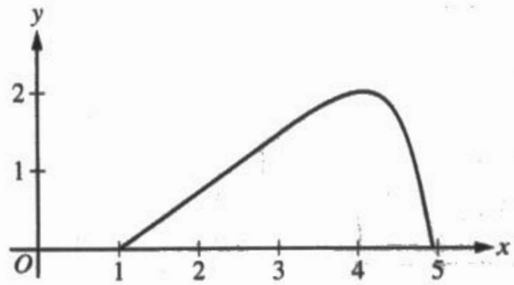
$1.6829 = 0.7357x - 0.7357 \Rightarrow x = 3.287$

Response for question 2(d)

$\frac{dy}{dx} = \frac{2\cos t}{e^{\cos t}}$

$\int_0^{\pi} \frac{2\cos t}{e^{\cos t}}\, dt = -3.551$

**Answer QUESTION 2 parts (a) and (b) on this page.**




Response for question 2(a)

$(x(t), y(t)) = (?, 2\sin t)$

$(x'(t), y'(t)) = (e^{\cos t}, 2\cos t)$

$(x''(t), y''(t) = (\cos t\, e^{\sin t}, 2 - \sin t)$

$\cancel{4}$ $\cancel{2}$

$t = 1$ ; $(.2329, -1.6829)$

Response for question 2(b)

$a_t = 1.5$

$2\sin t = 1.5$

$\sin t = .75$

$t = .85$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

**Answer QUESTION 2 parts (c) and (d) on this page.**

Response for question 2(c)

~~1.6~~

$e^{\cos t}$ $2\cos t$

$$\frac{1.0806}{1.7165} = \frac{y}{x} = m$$

$\boxed{.62953}$

Response for question 2(d)

$x'(t) = e^{\cos(t)}$

$x(t) = ?$

# Question 2

**Note:** Student samples are quoted verbatim and may contain spelling and grammatical errors.

## Overview

In this problem students were told that a particle is moving along a curve so that its position at time $t$ is $(x(t), y(t))$, with $y(t) = 2\sin t$, $\frac{dx}{dt} = e^{\cos t}$, and $0 \le t \le \pi$. Students were also told that at time $t = 0$, the particle is at position $(1, 0)$.

In part (a) students were asked to find the acceleration vector of the particle at time $t = 1$. This requires using a calculator to find the values $\left.\frac{d^2x}{dt^2}\right|_{t=1} = -1.444$ and $\left.\frac{d^2y}{dt^2}\right|_{t=1} = -1.683$.

In part (b) students were asked to find the first time $t$ at which the speed of the particle is $1.5$. A correct response will show the setup $\sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2} = 1.5$ and then use a calculator to find the first time $t$ in $[0, \pi]$ that satisfies this equation ($t = 1.254$).

In part (c) students were asked to find the slope of the line tangent to the particle's path at time $t = 1$ and then to find the position of the particle at this time. A correct response will indicate that the slope of the line tangent to the particle's path is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, then will use a calculator to find $\left.\frac{dy}{dx}\right|_{t=1} = 0.630$. The response will continue by noting that the $x$-coordinate of the position of the particle at time $t = 1$ is $x(0) + \int_0^1 \frac{dx}{dt}\,dt$ and will use a calculator to find that this value is $3.342$.

In part (d) students were asked to find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. A correct response will show the calculator setup of the integral of the particle's speed over this time interval, then evaluate the integral to find a total distance of $6.035$.

**Sample: 2A**
**Score: 8**

The response earned 8 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d).

In part (a) the response earned both points with the last two lines.

In part (b) the response earned the first point with the second line and earned the second point with the last line.

In part (c) the response did not earn the first point due to an incorrect evaluation of a correct derivative expression. The response earned the second point in the second line and earned the third point in the last line.

In part (d) the response earned the first point with the first line and earned the second point with the last line.

## Question 2 (continued)

**Sample: 2B**
**Score: 5**

The response earned 5 points: 2 points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d).

In part (a) the setup for both second derivatives occurs in the first two lines. The response earned both points with the work in the third line. Note also that the last line gives the correct decimal approximations.

In part (b) the response earned the first point with the first line. The response earned the second point with the second line.

In part (c) the response earned the first point with the second line. The response did not earn the second point because no definite integral is presented. The response did not earn the third point due to an incorrect value in the last line. (Note that the response attempts to approximate the position of $x$ at time $t = 1$ using a tangent line instead of finding the exact position using an integral.)

In part (d) the response did not earn the first point due to an incorrect integrand. The response did not earn the second point due to an incorrect value.

**Sample: 2C**
**Score: 2**

The response earned 2 points: 1 point in part (a), no points in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response did not earn the first point because it presents an incorrect value of $x''(1)$. The response earned the second point with a correct setup and value of $y''(1)$.

In part (b) the response did not earn the first point because it presents an incorrect equation. The response did not earn the second point because it presents an incorrect solution for $t$.

In part (c) the response earned the first point with the work presented. The response did not earn any further points because no additional work is presented.

In part (d) the response did not earn the first point because there is no integral presented. The response did not earn the second point because it does not present a value for the total distance traveled.

2023

# AP® Calculus BC

## Sample Student Responses and Scoring Commentary

### Inside:

Free-Response Question 3

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

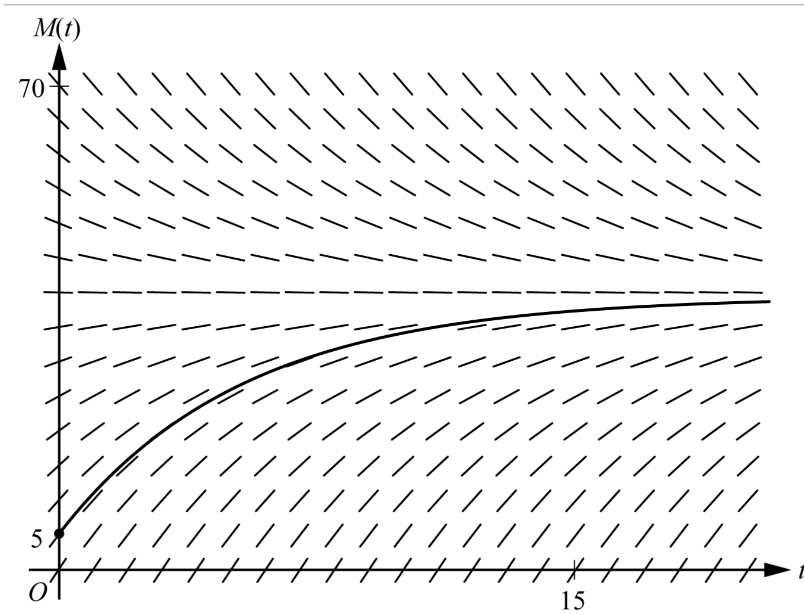
## Part B (AB or BC): Graphing calculator not allowed
## Question 3 — 9 points

**General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function $M$ models the temperature of the milk at time $t$, where $M(t)$ is measured in degrees Celsius ($°\text{C}$) and $t$ is the number of minutes since the bottle was placed in the pan. $M$ satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is $5°\text{C}$. It can be shown that $M(t) < 40$ for all values of $t$.

| | Model Solution | Scoring | |
|---|---|---|---|
| **(a)** | A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$. | | |
| |   | Solution curve | **1 point** |

**Scoring notes:**

- The solution curve must pass through the point $(0, 5)$, extend reasonably close to the left and right edges of the rectangle, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must lie entirely below the horizontal line segments at $M = 40$.

**Total for part (a) — 1 point**

**(b)** Use the line tangent to the graph of $M$ at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

| | | |
|---|---|---|
| $\left.\dfrac{dM}{dt}\right\|_{t=0} = \dfrac{1}{4}(40 - 5) = \dfrac{35}{4}$ | $\left.\dfrac{dM}{dt}\right\|_{t=0}$ | **1 point** |
| The tangent line equation is $y = 5 + \dfrac{35}{4}(t - 0)$.<br><br>$M(2) \approx 5 + \dfrac{35}{4} \cdot 2 = 22.5$<br><br>The temperature of the milk at time $t = 2$ minutes is approximately $22.5^\circ$ Celsius. | Approximation | **1 point** |

**Scoring notes:**

- The value of the slope may appear in a tangent line equation or approximation.
- A response of $5 + \frac{35}{4} \cdot 2$ is the minimal response to earn both points.
- A response of $\frac{1}{4}(40 - 5)$ earns the first point. If there are any subsequent errors in simplification, the response does not earn the second point.
- In order to earn the second point the response must present an approximation found by using a tangent line that:
  - passes through the point $(0, 5)$ and
  - has slope $\frac{35}{4}$ or a nonzero slope that is declared to be the value of $\frac{dM}{dt}$.
- An unsupported approximation does not earn the second point.
- The approximation need not be simplified, but the response does not earn the second point if the approximation is simplified incorrectly.

**Total for part (b) 2 points**

**(c)** Write an expression for $\frac{d^2M}{dt^2}$ in terms of $M$. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

| | | |
|---|---|---|
| $\dfrac{d^2M}{dt^2} = -\dfrac{1}{4}\dfrac{dM}{dt} = -\dfrac{1}{4}\left(\dfrac{1}{4}(40 - M)\right) = -\dfrac{1}{16}(40 - M)$ | $\dfrac{d^2M}{dt^2}$ | **1 point** |
| Because $M(t) < 40$, $\dfrac{d^2M}{dt^2} < 0$, so the graph of $M$ is concave down. Therefore, the tangent line approximation of $M(2)$ is an overestimate. | Overestimate with reason | **1 point** |

**Scoring notes:**

- The first point is earned for either $\frac{d^2M}{dt^2} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right)$ or $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$ (or equivalent). A response that presents any subsequent simplification error does not earn the second point.
- A response that presents an expression for $\frac{d^2M}{dt^2}$ in terms of $\frac{dM}{dt}$ but fails to continue to an expression in terms of $M$ (i.e., $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt}$) does not earn the first point but is eligible for the second point.
- If the response presents an expression for $\frac{d^2M}{dt^2}$ that is incorrect, the response is eligible for the second point only if the expression is a nonconstant linear function that is negative for $5 < M < 40$.
    - Special case: A response that presents $\frac{d^2M}{dt^2} = \frac{1}{16}(40 - M)$ does not earn the first point but is eligible to earn the second point for a consistent answer and reason.
- To earn the second point a response must include $\frac{d^2M}{dt^2} < 0$, or $\frac{dM}{dt}$ is decreasing, or the graph of $M$ is concave down, as well as the conclusion that the approximation is an overestimate.
- A response that presents an argument based on $\frac{d^2M}{dt^2}$ or concavity at a single point does not earn the second point.

**Total for part (c) 2 points**

---

**(d)** Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

| | | |
|---|---|---|
| $\frac{dM}{40 - M} = \frac{1}{4}dt$<br>$\int \frac{dM}{40 - M} = \int \frac{1}{4}\,dt$ | Separates variables | **1 point** |
| $-\ln\lvert 40 - M\rvert = \frac{1}{4}t + C$ | Finds antiderivatives | **1 point** |
| $-\ln\lvert 40 - 5\rvert = 0 + C \Rightarrow C = -\ln 35$<br>$M(t) < 40 \Rightarrow 40 - M > 0 \Rightarrow \lvert 40 - M\rvert = 40 - M$<br>$-\ln(40 - M) = \frac{1}{4}t - \ln 35$<br>$\ln(40 - M) = -\frac{1}{4}t + \ln 35$ | Constant of integration and uses initial condition | **1 point** |

| | | |
|---|---|---|
| $40 - M = 35e^{-t/4}$<br>$M = 40 - 35e^{-t/4}$ | Solves for $M$ | **1 point** |

**Scoring notes:**

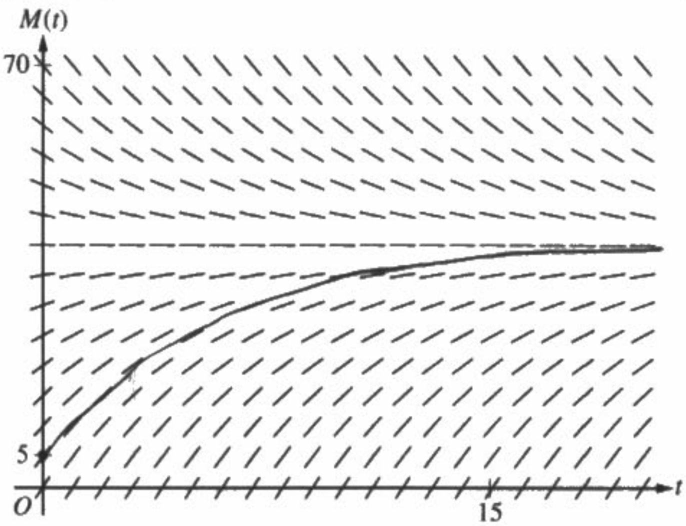
- A response with no separation of variables earns 0 out of 4 points.
- A response that presents an antiderivative of $-\ln(40 - M)$ without absolute value symbols is eligible for all 4 points.
- A response with no constant of integration can earn at most the first 2 points.
- A response is eligible for the third point only if it has earned the first 2 points.
  - Special Case: A response that presents $+\ln(40 - M) = \frac{t}{4} + C$ (or equivalent) does not earn the second point, is eligible for the third point, but not eligible for the fourth.
- An eligible response earns the third point by correctly including the constant of integration in an equation and substituting $0$ for $t$ and $5$ for $M$.
- A response is eligible for the fourth point only if it has earned the first 3 points.
- A response earns the fourth point only for an answer of $M = 40 - 35e^{-t/4}$ or equivalent.

**Total for part (d) 4 points**

**Total for question 3 9 points**

**Answer QUESTION 3 parts (a) and (b) on this page.**

Response for question 3(a)




Response for question 3(b)

$$\left.\frac{dM}{dt}\right|_{t=0} = \frac{1}{4}(40-5) = \frac{35}{4}$$

$$(M-5) = \frac{35}{4}(t-0)$$

$$M = \frac{35}{4}t + 5$$

$$M(2) \approx \frac{35}{4}\cdot 2 + 5$$

$$= \frac{35}{2} + 5$$

$$= \boxed{\frac{45}{2}\,°C}$$

**Answer QUESTION 3 parts (c) and (d) on this page.**

Response for question 3(c)

$$\frac{d^2M}{dt^2} = \frac{d}{dt}\left[\frac{1}{4}(40-M)\right]$$

$$= \frac{1}{4}\left(0 - \frac{dM}{dt}\right)$$

$$= \boxed{-\frac{1}{16}(40-M)}$$


Since $\frac{d^2M}{dt^2}$ value is always negative for all $t$ ($\because M(t) < 40$ for all $t$), the graph is always concave down. Thus, the approximation is overestimate.

Response for question 3(d)

$$\int \frac{1}{40-M}\, dM = \int \frac{1}{4}\, dt$$

$$-\ln|40-M| = \frac{1}{4}t + C$$

$$40 - M = Ce^{-\frac{1}{4}t}$$

$$M = ce^{-\frac{1}{4}t} + 40$$

$$5 = C \cdot e^{0} + 40$$

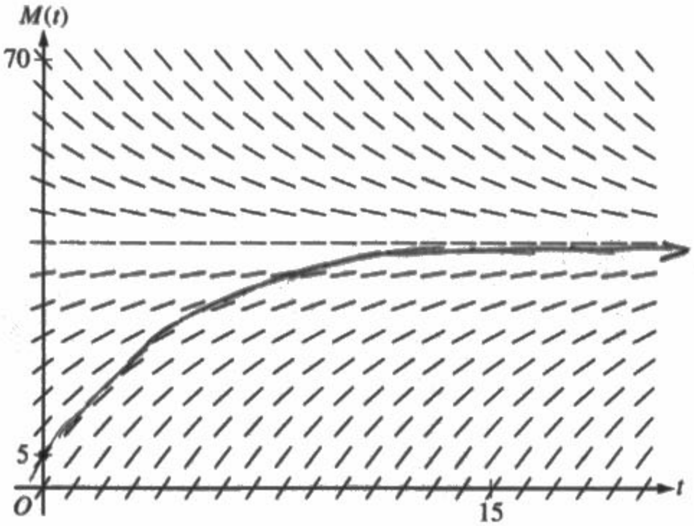
$$\rightarrow C = -35$$

$$\therefore M = -35e^{-\frac{1}{4}t} + 40$$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

**Answer QUESTION 3 parts (a) and (b) on this page.**

Response for question 3(a)




Response for question 3(b)

$\frac{dM}{dt} = \frac{1}{4}(40-M) \rightarrow \frac{1}{4}(40-5) = \frac{35}{4}$ ↙ slope

$M=5$ at $t=0$

$5 + \frac{70}{4} = \boxed{\frac{90}{4}\ C^\circ}$

$5 + (\frac{35}{4} \cdot 2) =$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5171/8

**Answer QUESTION 3 parts (c) and (d) on this page.**

Response for question 3(c)

c) $\frac{d^2M}{dt^2} = \frac{-M}{4} \rightarrow (-)$ + therefore $M(x)$ is always concave down

$M(2)$ from part (b) is an over estimation bc actual $M(2)$ is less than $\frac{90}{4}C°$ ⇒ $M(x)$ is concave down and $M'(x)$ is getting closer + closer to 0. Therefore, a tangent line won't account for the ever decreasing nature of $M'(x)$ + will overestimate

Response for question 3(d)

$u = 40 - M$
$-du = dM$

$$\int \frac{4}{40-M}\,dM = \int dt$$

$-\ln|40-M| + c = t \rightarrow e^{-\ln|40-5| + c} = e^0$

$-35 + e^c = 1$

$\ln e^c = \ln 36$

$c = \ln 36$

$$\boxed{t = -\ln|40-M| + \ln 36}$$

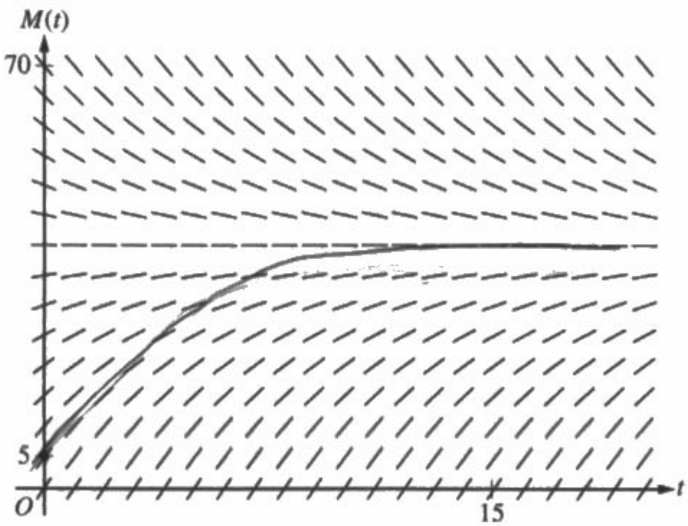
3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (a) and (b) on this page.

Response for question 3(a)

$\frac{1}{4}(40-5)$

10



Response for question 3(b)

$t$ = mins since bottle placed

$M(t)$ = milk temp at time $t$

$\frac{dM}{dt} = \frac{1}{4}(40-M)$

intial $t = 5°C$

$f'(0) = \frac{1}{4}(40-M)$

$= \frac{1}{4}(40-5)$

$= \frac{35}{4}$

$f(a) + f'(a)(x-a)$

$f(0) + f'(0)(2-0) \approx M(2)$

$5 + \frac{35}{4}(2)$

$5 + \frac{35}{2}$

$\frac{10}{2} + \frac{35}{2}$

$\frac{45}{2}$

$23.5\,°C \approx$ temperature of milk at $t = 2$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5178/8

**Answer QUESTION 3 parts (c) and (d) on this page.**

Response for question 3(c)

$$\frac{40-M}{4} \to \frac{4\left(-\frac{dM}{dt}\right)-(40-M)(0)}{16}$$

$$\frac{4\left(-\frac{dM}{dt}\right)}{16} = \frac{4\left(\frac{40-M}{4}\right)}{16} = \frac{40-M}{16} = \frac{d^2M}{dt^2}$$

$\frac{40-M}{16}$ at $t=2$

is positive at $t=2$

thus, the 2nd derivative / the graph is concave down meaning that it is an overestimate for the actual value of $M(2)$

Response for question 3(d)

$M(t)$

$$\frac{1}{4}(40-M) = \frac{dM}{dt}$$

$$40-M = \frac{4\,dM}{dt}$$

$$dt(40-M) = 4\,dM$$

$$dt = \frac{4\,dM}{40-M}$$

$$dt = dM \cdot \frac{1}{10-M}$$

$$t = -\ln(10-M) + C$$

$$0 = -\ln(10-5) + C$$

$$0 = \ln 5 + C$$

$$-\ln 5 = C$$

$$t = -\ln(10-M) - \ln 5$$

## Question 3

**Note:** Student samples are quoted verbatim and may contain spelling and grammatical errors.

### Overview

In this problem students were told that an increasing function $M$ models the temperature of a bottle of milk taken out of the refrigerator and placed in a pan of hot water to warm. The function $M$ satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M),$ where $t$ is measured in minutes since the bottle was placed in the pan and $M$ is measured in degrees Celsius. At time $t = 0,$ the temperature of the milk is $5°\text{C}.$

In part (a) students were given a slope field for the differential equation and asked to sketch the solution curve through the point $(0, 5).$ A correct response will draw a curve that passes through the point $(0, 5),$ follows the indicated slope segments, extends reasonably close to the left and right edges of the slope field, and lies entirely below the horizontal line segments at $M = 40.$

In part (b) students were asked to use the line tangent to the graph of $M$ at $t = 0$ to approximate $M(2).$ A correct response will find the slope of the tangent line when $t = 0$ is $\left.\frac{dM}{dt}\right|_{t=0} = \frac{35}{4}$ and then use the tangent line equation, $y = 5 + \frac{35}{4}t,$ to find that $M(2) \approx 22.5.$

In part (c) students were asked to find an expression for $\frac{d^2M}{dt^2}$ in terms of $M$ and then to use $\frac{d^2M}{dt^2}$ to reason whether the approximation from part (b) is an underestimate or overestimate for the actual value of $M(2).$ A correct response will differentiate the given differential equation to obtain $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{16}(40 - M),$ then use the information that $M(t) < 40$ to determine that the second derivative of $M$ is negative and therefore the graph of $M$ is concave up and the approximation in part (b) is an overestimate.

In part (d) students were asked to use separation of variables to find an expression for the particular solution to the given differential equation with initial condition $M(0) = 5.$ A correct response will separate the variables, integrate, use the initial condition to find the constant of integration, and arrive at a solution of $M = 40 - 35e^{-t/4}.$

**Sample: 3A**
**Score: 9**

The response earned 9 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 4 points in part (d).

In part (a) the response earned the point for the solution curve.

In part (b) the response earned the first point by stating $\frac{1}{4}(40 - 5)$ on the first line. The response correctly simplifies this expression, but this simplification is not needed. The second point was earned for the expression $\frac{35}{4} \cdot 2 + 5$ on the fourth line. The response simplifies the expression correctly as $\frac{45}{2},$ however, this is not needed to earn the second point.

## Question 3 (continued)

In part (c) the response earned the first point for the boxed expression $-\frac{1}{16}(40 - M)$ on the third line on the left. The second point was earned for the explanation given on the fourth, fifth, and sixth lines.

In part (d) the response earned the first point for the correct separation on the first line. The second point was earned for the correct antiderivatives on the second line. The third point was earned on the fifth line on the left for the equation $5 = C \cdot e^0 + 40.$ The fourth point was earned for solving for $M$ on the seventh line.

**Sample: 3B**
**Score: 6**

The response earned 6 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 2 points in part (d).

In part (a) the response earned the point for the solution curve.

In part (b) the response earned the first point by stating $\frac{1}{4}(40 - 5)$ on the first line. The response then correctly simplifies this expression to $\frac{35}{4},$ but this simplification is not needed. The second point was earned for the expression $5 + \frac{70}{4}$ on the third line. The boxed expression, though correct, is not needed to earn the second point.

In part (c) the response did not earn the first point because the expression for $\frac{d^2M}{dt^2}$ is not correct. The response is eligible for the second point because the expression for $\frac{d^2M}{dt^2}$ is a nonconstant linear function that is negative for $5 < M < 40.$ The second point was earned for the statement on the first through third lines. On the third through sixth lines, the response makes statements that are correct but not needed to earn the second point.

In part (d) the response earned the first point for the correct separation on the third line on the left. The second point was not earned because the antiderivative of $\frac{4}{40 - M}$ is not correct. However, the response is eligible to earn the third point. The third point was earned on the first line on the right for the equation $-\ln|40 - 5| + C = 0.$ The response is not eligible for the fourth point because the response did not earn all of the first three points in this part.

**Sample: 3C**
**Score: 2**

The response earned 2 points: no points in part (a), 1 point in part (b), no points in part (c), and 1 point in part (d).

In part (a) the response did not earn the point because the solution curve does not lie entirely below the horizontal segments.

In part (b) the response earned the first point for the expression $\frac{1}{4}(40 - 5)$ on the sixth line on the left. The response did not earn the second point because the final approximation is not simplified correctly on the seventh line on the right.

## Question 3 (continued)

In part (c) the response did not earn the first point because the expression for $\frac{d^2M}{dt^2}$ is not correct. The response provides a form of the only positive second derivative that is eligible for the second point, but it did not earn the second point because both an incorrect conclusion and local argument are made.

In part (d) the response earned the first point for the correct separation in the fifth line on the left. The second point was not earned because the antiderivative is not correct due to a simplification error prior to integration. The presented antiderivative is not eligible to earn the third or fourth points.

2023

# AP® Calculus BC

## Sample Student Responses and Scoring Commentary

**Inside:**

Free-Response Question 4

- ☑ Scoring Guidelines
- ☑ Student Samples
- ☑ Scoring Commentary

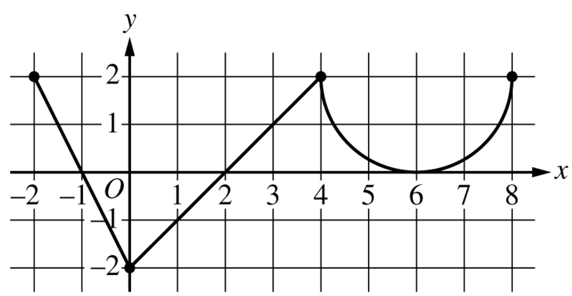
## Part B (AB or BC): Graphing calculator not allowed

## Question 4 — 9 points

**General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.




Graph of $f'$

The function $f$ is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of $f'$, the derivative of $f$, consists of two line segments and a semicircle, as shown in the figure.

| | Model Solution | Scoring | |
|---|---|---|---|
| **(a)** | Does $f$ have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer. | | |
| | $f'(x) > 0$ on $(2, 6)$ and $f'(x) > 0$ on $(6, 8)$.<br><br>$f'(x)$ does not change sign at $x = 6$, so there is neither a relative maximum nor a relative minimum at this location. | Answer with reason | **1 point** |

**Scoring notes:**

- A response that declares $f'(x)$ does not change sign at $x = 6$, so neither, is sufficient to earn the point.
- A response does not have to present intervals on which $f'(x)$ is positive or negative, but if any are given, they must be correct. Any presented intervals may include none, one, or both endpoints.
- A response that declares $f'(x) > 0$ before and after $x = 6$ does not earn the point.

**Total for part (a) — 1 point**

**(b)** On what open intervals, if any, is the graph of $f$ concave down? Give a reason for your answer.

| | | |
|---|---|---|
| The graph of $f$ is concave down on $(-2, 0)$ and $(4, 6)$ because $f'$ is decreasing on these intervals. | Intervals | **1 point** |
| | Reason | **1 point** |

**Scoring notes:**

- The first point is earned only by an answer of $(-2, 0)$ and $(4, 6)$, or these intervals including one or both endpoints.
- A response must earn the first point to be eligible for second point.
- To earn the second point a response must correctly discuss the behavior of $f'$ or the slopes of $f'$.
- Special case: A response that presents exactly one of the two correct intervals with a correct reason earns 1 out of 2 points.

**Total for part (b) 2 points**

**(c)** Find the value of $\lim_{x\to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

| | | |
|---|---|---|
| Because $f$ is differentiable at $x = 2$, $f$ is continuous at $x = 2$, so $\lim_{x\to 2} f(x) = f(2) = 1$. $\lim_{x\to 2}(6f(x) - 3x) = 6\cdot 1 - 3\cdot 2 = 0$ $\lim_{x\to 2}\left(x^2 - 5x + 6\right) = 0$ | Limits of numerator and denominator | **1 point** |
| Because $\lim_{x\to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$ is of indeterminate form $\dfrac{0}{0}$, L'Hospital's Rule can be applied. Using L'Hospital's Rule, $\lim_{x\to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x\to 2} \dfrac{6f'(x) - 3}{2x - 5} = \dfrac{6\cdot 0 - 3}{2\cdot 2 - 5} = 3.$ | Uses L'Hospital's Rule | **1 point** |
| | Answer | **1 point** |

**Scoring notes:**

- The first point is earned by the presentation of two separate limits for the numerator and denominator.
- A response that presents a limit explicitly equal to $\dfrac{0}{0}$ does not earn the first point.
- The second point is earned by applying L'Hospital's Rule, that is, by presenting at least one correct derivative in the limit of a ratio of derivatives.
- The third point is earned for the correct answer with supporting work.

**Total for part (c) 3 points**

**(d)** Find the absolute minimum value of $f$ on the closed interval $[-2, 8]$. Justify your answer.

| | | |
|---|---|---|
| $f'(x) = 0 \Rightarrow x = -1,\ x = 2,\ x = 6$ | Considers $f'(x) = 0$ | **1 point** |
| The function $f$ is continuous on $[-2, 8]$, so the candidates for the location of an absolute minimum for $f$ are $x = -2$, $x = -1$, $x = 2$, $x = 6$, and $x = 8$. | Justification | **1 point** |
| | Answer | **1 point** |

| $x$ | $f(x)$ |
|---|---|
| $-2$ | $3$ |
| $-1$ | $4$ |
| $2$ | $1$ |
| $6$ | $7 - \pi$ |
| $8$ | $11 - 2\pi$ |

The absolute minimum value of $f$ is $f(2) = 1$.

**Scoring notes:**

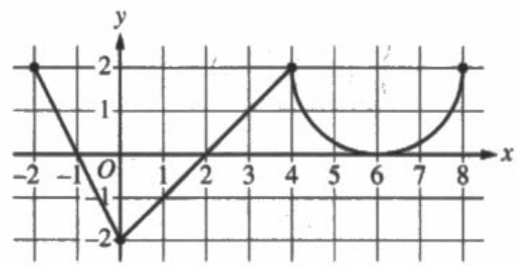
- To earn the first point a response must state $f' = 0$ or equivalent. Listing the zeros of $f'$ is not sufficient.
- A response that presents any error in evaluating $f$ at any critical point or endpoint will not earn the justification point.
- A response need not present the value of $f(-1)$ provided $x = -1$ is eliminated because it is the location of a local maximum. A response need not present the value of $f(6)$ provided $x = 6$ is eliminated by reference to part (a) or eliminated through analysis.
- A response need not present the value of $f(8)$ provided there is a presentation that argues $f'(x) \geq 0$ for $x > 2$ and, therefore, $f(8) > f(2)$.
- A response that does not consider both endpoints does not earn the justification point.
- The answer point is earned only for indicating that the minimum value is $1$. It is not earned for noting that the minimum occurs at $x = 2$.

**Total for part (d) 3 points**

**Total for question 4 9 points**

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

**Answer QUESTION 4 parts (a) and (b) on this page.**




Graph of $f'$

Response for question 4(a)

f has neither a relative max. or min. at $x=6$ since there is no sign change on $f'$ at $x=6$.

Response for question 4(b)

f is concave down on $(-2,0)$ and $(4,6)$ because $f'(x)$ is decreasing on those intervals



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5216/10

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

**Answer QUESTION 4 parts (c) and (d) on this page.**

Response for question 4(c)

$\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6}$

$\lim_{x\to 2} 6f(x)-3x = 0$

$\lim_{x\to 2} x^2-5x+6 = 0$

Since f is differatiable and therefore contious at x=2

L'Hospital's Rule applys

$\lim_{x\to 2} \frac{6f'(x)-3}{2x-5} = \frac{-3}{-1} = \boxed{3}$

Response for question 4(d)

$1-$ $\frac{1}{4}\pi 4$ $\pi$ $4-\pi$ $\frac{1}{2}\pi r^2$ $2\pi$ $8-2\pi$

Critical values

$f'(x)=0$ at.

$-1, 2, 6$

| x | f(x) |
|---|---|
| -2 | $1-\int_{-2}^{2} f(x)\,dx = 1-(-2) = 3$ |
| -1 | $1-\int_{-1}^{2} f(x)\,dx = 1--3 = 4$ |
| 2 | 1 |
| 6 | $1+\int_{2}^{6} f(x)\,dx = 1+2+(4-\pi) = 7-\pi$ |
| 8 | $1+\int_{2}^{8} f(x)\,dx = 1+2+(8-2\pi) = 11-2\pi$ |

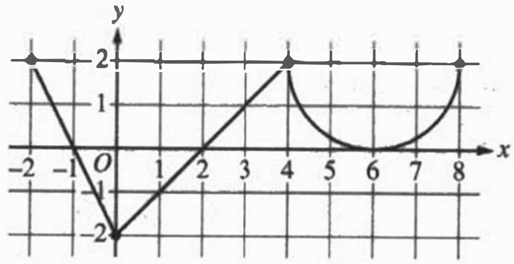
The absolute min value of f on [-2, 8] is at x=2 and is equal to 1.



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

**Answer QUESTION 4 parts (a) and (b) on this page.**




Graph of $f'$

**Response for question 4(a)**

At $x=6$, $f$ does not have relative minimum nor relative maximum because $f''$ does not change sign, meaning $f$ does not change from ~~that~~ increasing to decreasing nor decreasing to increasing.

**Response for question 4(b)**

$f$ is concave down on the interval $(-2,0) \cup (4,6)$ because $f'$ is decreasing, meaning $f''$ is negative.



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0015407

Q5171/10

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

**Answer QUESTION 4 parts (c) and (d) on this page.**

Response for question 4(c)

$$\lim_{x\to 2}\frac{6f(x)-3x}{x^2-5x+6}=\frac{6f(2)-3\times 2}{2^2-5\times 2+6}=\frac{6\times 1-6}{4-10+6}=\frac{0}{0}$$

$$\lim_{x\to 2}\frac{6f'(x)-3}{2x-5}=\frac{6f'(2)-3}{2\times 2-5}=\frac{6\times 0-3}{4-5}=\frac{-3}{-1}=3$$

Response for question 4(d)

The absolute minimum value is at $x=2$ because $f'$ changes from negative to positive.

$f(2)=1$ The absolute minimum value is 1.

Also, $f(-2)=f(0)$ because the positive area and negative area canceled out, and $f$ kept dereasing on $(0,2)$ according to the negative $f'$ on the interval, so absolute.

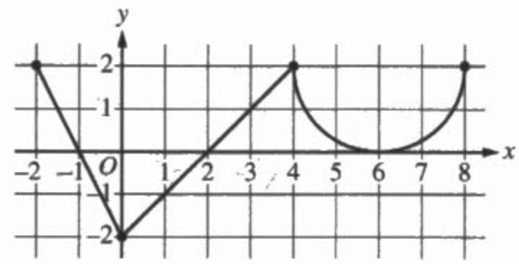


Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5171/11

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

**Answer QUESTION 4 parts (a) and (b) on this page.**




Graph of $f'$

$f(2) = 1$

---

Response for question 4(a)

Neither. Since $f'(6)=0$, $f$ does have an extrema at $x=6$, but since $f''(6)=0$, as well, $f(6)$ ~~is~~ must be an inflection point.

---

Response for question 4(b)

For $f(x)$ to be concave down, $f''(x)$ must be less than 0. Based on the given graph of $f'$, the graph of $f$ is concave down on the intervals $-2<x<0$ and $4<x<6$ because on these intervals, $f''(x)<0$.

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

**Answer QUESTION 4 parts (c) and (d) on this page.**

Response for question 4(c)

$f(x) = x-2,\ 0 \le x \le 4$

$\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \lim_{x\to 2} \frac{3(2f(x)-x)}{(x-3)(x-2)} \rightarrow \frac{2f'(2)}{(x-3)} = 0$

Response for question 4(d)

Min $\rightarrow f'(x)=0,\ f''(x)>0$

$f'(2)=0$

$f''(2)=1 \rightarrow 1>0$

$f(2)=1$

$\boxed{f(-2)=-1}$

Endpoints

$f(-2) = f(2) - \int_{-2}^{2} f'(x)\,dx$

$= 1-(3-1)$

$= 1-2$

$= -1$

$A = \frac{\pi r^2}{2}$

$A_{sc} = \frac{4\pi}{2} = 2\pi$

$f(8) = f(2) + \int_{2}^{8} f'(x)\,dx$

$= 1 + (2 + (8-2\pi)) = 11-2\pi$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

## Question 4

**Note:** Student samples are quoted verbatim and may contain spelling and grammatical errors.

### Overview

In this problem the graph of $f'$, which consists of a semicircle and two line segments on the interval $-2 \le x \le 8$, is provided. It is given that $f$ is defined on the closed interval $[-2, 8]$, and that $f(2) = 1$.

In part (a) students were asked to reason whether $f$ has a relative minimum, relative maximum, or neither at $x = 6$. A correct response will use the given graph to reason that $f'$ does not change signs at $x = 6$, although $f'(6) = 0$. Therefore $f$ has neither a relative maximum nor a relative minimum at this point.

In part (b) students were asked to find all open intervals where $f$ is concave down. A correct response will reason that a function is concave down when its first derivative is decreasing, and therefore $f$ is concave down on the intervals $(-2, 0)$ and $(4, 6)$.

In part (c) students were asked to find $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$ or to show that this limit does not exist (and to justify their answer). A correct response will report that as $x$ approaches $2$, both the numerator and denominator of this ratio approach $0$, and so L'Hospital's Rule applies. Using L'Hospital's Rule, the limit is shown to be $3$.

In part (d) students were asked to find the absolute minimum value of $f$ on the closed interval $[-2, 8]$ with a justification for their answer. A correct response will indicate the possible candidates for the location of the absolute minimum are the interval endpoints and the critical points $x = -1$, $x = 2$, and $x = 6$. A response could then reference work from part (a) to eliminate $x = 6$ as the location of a relative (or absolute) minimum, and could use the fact that the graph of $f'$ changes from positive to negative at $x = -1$ in order to argue that a relative maximum occurs at $x = -1$. In addition, the given graph of $f'$ indicates that $f'(x) \ge 0$ for $x > 2$, so the endpoint $x = 8$ cannot be the location of the absolute minimum value. The value of $f(2)$ is given in the stem of the problem, and using geometry, $f(-2) = 1 + \int_2^{-2} f'(x)\, dx = 3$. Therefore, the absolute minimum value of $f$ on this closed interval is $f(2) = 1$. (Alternatively, a response could evaluate the function $f$ at each of these five points and conclude that the absolute minimum is $f(2) = 1$.)

**Sample: 4A**
**Score: 9**

The response earned 9 points: 1 point in part (a), 2 points in part (b), 3 points in part (c), and 3 points in part (d).

In part (a) the response earned the point with a correct conclusion of no relative maximum or minimum and correct reasoning that there is no sign change for $f'(x)$ at $x = 6$.

In part (b) the response earned 2 points. The first point was earned with correct presentation of the intervals of concavity. The second point was earned with correct reasoning that $f'(x)$ is decreasing on these intervals.

In part (c) the response earned 3 points. The first point was earned with correct presentation of limits of the numerator and denominator. The second point was earned because the ratio of derivatives presented is correct. The third point was earned with a correct answer.

## Question 4 (continued)

In part (d) the response earned 3 points. The first point was earned with the consideration of $f'(x) = 0.$ The response earned the second point with a correct analysis with a Candidates Test. The response earned the third point with a correct answer of 1.

**Sample: 4B**
**Score: 6**

The response earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the point with a correct conclusion of no relative maximum or minimum and correct reasoning that $f'(x)$ does not change signs at $x = 6.$

In part (b) the response earned 2 points. The first point was earned with correct presentation of the intervals of concavity. The second point was earned with correct reasoning that $f'(x)$ is decreasing.

In part (c) the response earned 2 points. The first point was not earned because the response does not present limits of the numerator and denominator. The second point was earned because the ratio of derivatives presented is correct. The third point was earned with a correct answer.

In part (d) the response earned 1 point. The first point was earned with the consideration of $f'(x)$ changing signs from negative to positive at $x = 2.$ The response did not earn the second point because there is no analysis with the endpoints or the elimination of interior points as possible minimums. The response is not eligible for the third point.

**Sample: 4C**
**Score: 2**

The response earned 2 points: no points in part (a), 1 point in part (b), no points in part (c), and 1 point in part (d).

In part (a) the response did not earn the point because the reasoning that there must be an inflection point at $f(6)$ is insufficient to earn the point.

In part (b) the response earned 1 point. The first point was earned with correct presentation of the intervals of concavity. The second point was not earned because the reasoning is based solely on $f''(x) < 0.$

In part (c) the response earned no points. The first point was not earned because the response does not present limits of the numerator and denominator. The second point was not earned because the ratio of derivatives presented is incorrect. The response is not eligible for the third point.

In part (d) the response earned 1 point. The first point was earned with the consideration of $f'(x) = 0$ in the first line. The response did not earn the second point because there is no analysis with the critical values $x = -1,$ $x = 2,$ and $x = 6.$ The response is ineligible for the third point.

2023

# AP® Calculus BC

## Sample Student Responses and Scoring Commentary

**Inside:**

Free-Response Question 5

☑ Scoring Guidelines

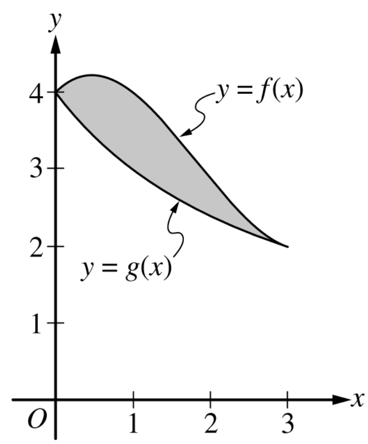
☑ Student Samples

☑ Scoring Commentary

## Part B (BC): Graphing calculator not allowed
## Question 5 — 9 points

**General Scoring Notes**
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.




The graphs of the functions $f$ and $g$ are shown in the figure for $0 \le x \le 3$. It is known that $g(x) = \frac{12}{3+x}$ for $x \ge 0$. The twice-differentiable function $f$, which is not explicitly given, satisfies $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

| Model Solution | Scoring | |
|---|---|---|
| **(a)** Find the area of the shaded region enclosed by the graphs of $f$ and $g$. | | |
| $\text{Area} = \int_0^3 (f(x) - g(x))\,dx = \int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx$ | Integrand | **1 point** |
| $= 10 - \int_0^3 \frac{12}{3+x}\,dx = 10 - 12\left[\ln\lvert 3+x\rvert\right]_0^3$ | Antiderivative of $g(x)$ | **1 point** |
| $= 10 - 12(\ln 6 - \ln 3) = 10 - 12(\ln 2)$ | Answer | **1 point** |

**Scoring notes:**

- The first point is earned for any of the integrands $f(x) - g(x)$, $g(x) - f(x)$, $\lvert f(x) - g(x)\rvert$, or $\lvert g(x) - f(x)\rvert$ in any definite integral. If the limits are incorrect, the response does not earn the third point.

- The first point is earned with an implied integrand for $f$ and explicit integrand for $g$, such as $10 - \int_0^3 g(x)\,dx$.
- The second point is earned for finding $a\int \frac{dx}{3+x} = a \cdot \ln|3+x|$ or $a \cdot \ln(3+x)$.
- A response is eligible for the third point only if it has earned the first 2 points. The third point is earned only for the correct answer. The answer does not need to be simplified; however, if simplification is attempted, it must be correct.
- A response is not eligible for the third point with incorrect limits of integration for $u$-substitution, for example, $\int_0^3 \frac{12}{3+x}\,dx = \int_0^3 \frac{12}{u}\,du = 12\left[\ln(x+3)\right]_0^3$.
- A response with incorrect communication, such as "Area $= \int_0^3 (g(x) - f(x))\,dx = 10 - 12(\ln 2)$," does not earn the third point. However, a response of "$\int_0^3 (g(x) - f(x))\,dx = 12(\ln 2) - 10,$ so the area is $10 - 12(\ln 2)$" earns all 3 points.

**Total for part (a) 3 points**

---

**(b)** Evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

| | | |
|---|---|---|
| $\int_0^\infty (g(x))^2\,dx = \lim\limits_{b\to\infty} \int_0^b \frac{144}{(3+x)^2}\,dx$ | Limit notation | **1 point** |
| $= \lim\limits_{b\to\infty} \left( -\frac{144}{(3+x)}\Bigg|_0^b \right)$ | Antiderivative | **1 point** |
| $= \lim\limits_{b\to\infty} \left( -\frac{144}{3+b} + \frac{144}{3} \right) = 48$ | Answer | **1 point** |

**Scoring notes:**

- To earn the first point a response must correctly use limit notation throughout the problem and not include arithmetic with infinity, for example, $\left[-\frac{144}{3+x}\right]_0^\infty$ or $-\frac{144}{3+\infty} + 48$.
- The second point can be earned by finding an antiderivative of the form $-\frac{a}{(3+x)}$ for $a > 0$, from an indefinite or improper integral, with or without correct limit notation. If $a \neq 144$, the response does not earn the third point.
- The third point is earned only for an answer of $48$ (or equivalent).
- A response is not eligible for the third point with incorrect limits of integration for $u$-substitution, for example, $\lim\limits_{b\to\infty} \int_0^b \frac{144}{u^2}\,du = \lim\limits_{b\to\infty} \left[-\frac{144}{3+x}\right]_0^b$.

**Total for part (b) 3 points**

**(c)** Let $h$ be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\, dx$.

| | | |
|---|---|---|
| Using integration by parts,<br>$u = x \qquad dv = f'(x)\, dx$<br>$du = dx \quad v = f(x)$ | $u$ and $dv$ | **1 point** |
| $\int h(x)\, dx = \int x \cdot f'(x)\, dx = x \cdot f(x) - \int f(x)\, dx$ | $\int h(x)\, dx$<br>$= x \cdot f(x) - \int f(x)\, dx$ | **1 point** |
| $\int_0^3 h(x)\, dx = \int_0^3 x \cdot f'(x)\, dx = x \cdot f(x)\Big\vert_0^3 - \int_0^3 f(x)\, dx$<br>$= (3 \cdot f(3) - 0 \cdot f(0)) - 10 = 3 \cdot 2 - 0 - 10 = -4$ | Answer | **1 point** |

**Scoring notes:**

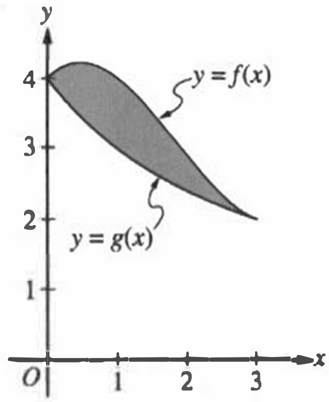
- The first and second points are earned with an implied $u$ and $dv$ in the presence of $x \cdot f(x) - \int f(x)\, dx$ or $x \cdot f(x)\Big\vert_0^3 - 10$.
- Limits of integration may be present, omitted, or partially present in the work for the first and second points.
- The tabular method may be used to show integration by parts. In this case, the first point is earned by having columns (labeled or unlabeled) that begin with $x$ and $f'(x)$. The second point is earned for $x \cdot f(x) - \int f(x)\, dx$.
- The third point is earned only for the correct answer and can only be earned if the first 2 points were earned.

**Total for part (c) 3 points**

**Total for question 5 9 points**

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

**Answer QUESTION 5 part (a) on this page.**




Response for question 5(a)

$$\int_0^3 f(x) - g(x)\, dx = \int_0^3 f(x)dx - \int_0^3 \frac{12}{3+x}\, dx$$

$$= 10 - \left(12 \ln |3+x|\right)\Big|_0^3$$

$$= \boxed{10 - (12\ln 6 - 12\ln 3)}.$$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

**Answer QUESTION 5 parts (b) and (c) on this page.**

Response for question 5(b)

b) $\int_0^\infty \left(\frac{12}{3+x}\right)^2 dx = 144\int_0^\infty (3+x)^{-2}\, dx$

$144 \cdot \lim_{a\to\infty} \int_0^a (3+x)^{-2}\, dx = 144 \cdot \lim_{a\to\infty} \left(-(3+x)^{-1}\right)\Big|_0^a$

$= 144 \cdot \lim_{a\to\infty} \left(-\frac{1}{3+x}\right)\Big|_0^a = 144 \cdot \lim_{a\to\infty} \left(\cancel{-\frac{1}{3+a}} - \left(-\frac{1}{3+0}\right)\right)$

$= 144 \cdot \frac{1}{3} = \boxed{48}$

Response for question 5(c)

$\int_0^3 h(x)\, dx = \int_0^3 x f'(x)\, dx$ $\qquad u = x \quad dv = f'(x)\,dx$

$du = dx \quad v = f(x)$

$= x \cdot f(x)\Big|_0^3 - \int_0^3 f(x)\, dx$

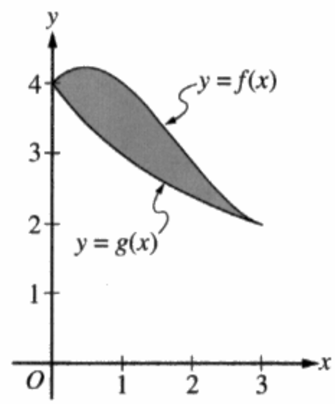
$= 3 \cdot 2 - 10 = \boxed{-4}$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

**Answer QUESTION 5 part (a) on this page.**




Response for question 5(a)

$$\text{area} = \underbrace{\int_0^3 f(x)\,dx}_{10} - \underbrace{\int_0^3 g(x)\,dx}_{\swarrow}$$

$$\left[12\ln|3+x|\right]\Big|_0^3 = \underline{12\ln 6 - 12\ln 3}$$

$$10 - 12\ln 6 + 12\ln 3$$

$$\boxed{\text{area} = 10 + 12(\ln 3 - \ln 6)}$$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5178/12

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

**Answer QUESTION 5 parts (b) and (c) on this page.**

Response for question 5(b)

$$\lim_{b\to\infty^-}\int_0^b \left(\frac{12}{3+x}\right)^2 dx \rightarrow \lim_{b\to\infty^-}\int_0^b \frac{144}{(3+x)^2}dx \qquad \begin{array}{l} u = 3+x \\ du = dx \end{array}$$

$$\lim_{b\to\infty^-}\int_0^b 144u^{-2}\,du$$

$$\lim_{b\to\infty^-}\left[-144u^{-1}\right]\Big|_0^b$$

$$\lim_{b\to\infty^-}\left[-\frac{144}{3+x}\right]\Big|_0^b$$

$$0+\frac{144}{3}$$

$$\boxed{\frac{144}{3}}$$

Response for question 5(c)

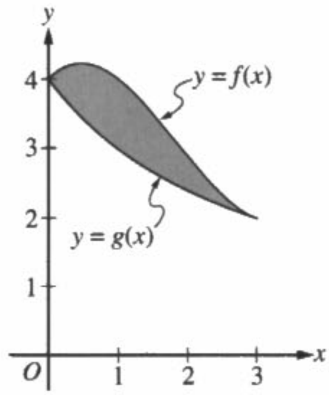
$$\int_0^3 x\cdot f''(x)\,dx \quad \cancel{\ldots}$$

$$x\cdot f(x) - \int_0^3 f'(x)\,dx$$

$$\left[x\cdot f(x) - f(x)\right]\Big|_0^3 = 2f(3)+f(0)$$

$$4+4=\boxed{8}$$

**Answer QUESTION 5 part (a) on this page.**




Response for question 5(a)

$\int_0^3 f(x) - g(x)\; dx = \text{Area}$

$\int_0^3 g(x)\,dx = \int_0^3 \frac{12}{3+x}\,dx = 12\ln|3+x|\Big|_0^3 = \cancel{12\ln|3| - 12\ln|6|}$

$12\int_0^3 \frac{12}{3+x}\,dx$

$\cancel{10 - 12\ln|6| = \text{Area}}$ $\quad [12\ln|3+3|] - [12\ln|3+0|]$

$\cancel{10 - 12\ln|6| = \text{Area}}$ $\quad 12\ln|6| - 12\ln|3| = \ln|3|$

$10 - \ln|3| = \text{Area}$

5 5 5 5 5 **NO CALCULATOR ALLOWED** 5 5 5 5 5

**Answer QUESTION 5 parts (b) and (c) on this page.**

**Response for question 5(b)**

$$\int_0^\infty (g(x))^2\,dx$$

$$\lim_{b\to\infty}\int_0^b (g(x))^2\,dx$$

$$\lim_{b\to\infty}\int_0^b \left(\frac{12}{3+x}\right)^2 dx$$

$$\frac{12}{3+x}\cdot\frac{12}{3+x} = \frac{144}{9+x^2+6x}$$

$$\lim_{b\to\infty}\int_0^{b} \frac{144}{x^2+6x+9}\,dx$$

$$\lim_{b\to\infty} 144\int_0^b \frac{1}{x^2+6x+9}\,dx$$

$$\lim_{b\to\infty} 144\ln|x^2+6x+9|\Big|_0^b = \left[144(b^2+6b+9)\right] - \left[144(0^2+6+9)\right] = \cancel{\infty}$$

$9$

$\cancel{144[\infty^2+6(\infty)+9]}$

$\ln|\infty^2+6(\infty)+9 = \ln|\infty| = \infty$

$\infty - (\ln|0^2+6(0)+9|)(144) = \infty \therefore$ divergent

since ~~the integral~~ $\infty = \infty$ in integral Test = divergent

**Response for question 5(c)**

$h(x) = x\cdot f'(x)$

$$\int_0^3 h(x)\,dx \qquad \int_0^3 x\cdot f'(x)\,dx$$

$\cancel{6x}$ $\cancel{21}$

$$x\int_0^3 f'(x)\,dx$$

$$x\,f(x)\Big|_0^3$$

$$[3(f(3))] - (0(f(0))) = 6$$

$f(3) = 2$, $f(0) = 0$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

# Question 5

**Note:** Student samples are quoted verbatim and may contain spelling and grammatical errors.

## Overview

In this problem students were given a figure showing a shaded region bounded by the graphs of functions $f$ and $g$ for $0 \le x \le 3$. Students were told that $g(x) = \dfrac{12}{3 + x}$ for $x \ge 0$ and that $f$ is differentiable with $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

In part (a) Students were asked to find the area of the shaded region. This requires setting up and evaluating $\int_0^3 (f(x) - g(x))\,dx$. To evaluate, a student will need to separate into two integrals, $\int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx$, and find an antiderivative for the function $g$. A correct response will provide an answer of $10 - 12(\ln|3 + x|)\Big|_0^3 = 10 - 12(\ln 6 - \ln 3)$.

In part (b) students were asked to evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$ or to show that the integral diverges. A correct response will employ correct limit notation to rewrite the improper integral with a variable upper limit, find the correct antiderivative ($\int \dfrac{144}{(3 + x)^2}\,dx = -\dfrac{144}{(3 + x)}$), and continue the correct limit notation to find a value of $48$.

In part (c) students were asked to find the value of $\int_0^3 h(x)\,dx$ given that $h(x) = x \cdot f'(x)$. A correct response will recognize the need to use integration by parts to find $\int_0^3 h(x)\,dx = x \cdot f(x)\Big|_0^3 - \int_0^3 f(x)\,dx = 6 - 0 - 10 = -4$.

**Sample: 5A**
**Score: 9**

The response earned 9 points: 3 points in part (a), 3 points in part (b), and 3 points in part (c).

In part (a) the response earned the first point with the correct definite integral at the start of line 1. The response earned the second point with the correct antiderivative of $g(x)$ in line 2. The response earned the third point with the correct boxed answer in line 3. Numerical simplification is not required.

In part (b) the response earned the first point with the correct use of limit notation in the expression on the left in line 2 and the consistent and correct use of the limiting process to the end of the response. The response earned the second point with the correct antiderivative on the right in line 2. The response would have earned the third point with the correct answer of $144 \cdot \dfrac{1}{3}$ in line 4. In this case, the response correctly simplifies to the answer of $48$ in line 4 and earned the third point.

## Question 5 (continued)

In part (c) the response earned the first point with the correct identification of $u$ and $dv$ in line 1 to the right. The response earned the second point with the correct application of integration by parts in line 2. The response would have earned the third point with the answer of $3 \cdot 2 - 10$ in line 3. In this case, the response correctly simplifies to the answer of $-4$ in line 3 and earned the third point.

**Sample: 5B**
**Score: 5**

The response earned 5 points: 3 points in part (a), 2 points in part (b), and no points in part (c).

In part (a) the response earned the first point with the difference of definite integrals in line 1. The response earned the second point with the correct antiderivative of $g(x)$ in line 2. The response would have earned the third point with the expression in line 3. In this case, the response correctly simplifies and earned the third point with the boxed answer in line 4.

In part (b) the response earned the first point with the correct use of limit notation in the expression on the left in line 1 and the consistent, correct use of the limiting process to the end of the response. The response earned the second point with the correct antiderivative of the $u$-substitution integrand in line 3. The response is not eligible for the third point because the response uses incorrect bounds of integration on the $u$-substitution integral in line 2.

In part (c) the response did not earn the first point because no expressions for $u$ and $dv$ have been clearly identified. The response did not earn the second point because the potential application of integration by parts on line 2 is incorrect. The response did not earn the third point because the answer is not correct.

**Sample: 5C**
**Score: 2**

The response earned 2 points: 2 points in part (a), no points in part (b), and no points in part (c).

In part (a) the response earned the first point with the correct definite integral in line 1. The response earned the second point with the correct antiderivative of $g(x)$ at the end of line 2. The response evaluates the integral of $g(x)$ correctly; however, the simplification at the end of line 4 is not correct. The response did not earn the third point because the answer is not correct.

In part (b) the response did not earn the first point. The response correctly uses limit notation in line 2; however, the response omits the limit in line 7 on the right and, thus, has not correctly used the limiting process for the entire response. The response did not earn the second point because the antiderivative in line 7 is not correct. The response did not earn the third point because the answer is not correct.

In part (c) the response did not earn the first, second, and third points because the response does not use integration by parts and the answer is not correct.

2023

# AP® Calculus BC

## Sample Student Responses and Scoring Commentary

**Inside:**

Free-Response Question 6

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

## Part B (BC): Graphing calculator not allowed
## Question 6 — 9 points

**General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The function $f$ has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

| Model Solution | Scoring | |
|---|---|---|

**(a)** Find $f^{(4)}(x)$, the fourth derivative of $f$ with respect to $x$. Write the fourth-degree Taylor polynomial for $f$ about $x = 0$. Show the work that leads to your answer.

| Model Solution | Scoring | |
|---|---|---|
| $f^{(4)}(x) = -2 \cdot f'\left(x^2\right) + (-2x) f''\left(x^2\right) \cdot 2x$ | Form of product rule | **1 point** |
| | $f^{(4)}(x)$ | **1 point** |
| $f''(0) = -f(0) = -2$<br>$f'''(0) = -2(0) \cdot f'(0) = 0$<br>$f^{(4)}(0) = -2 \cdot f'(0) + 0 \cdot f''(0) \cdot 0 = -2 \cdot 3 + 0 = -6$<br><br>The fourth-degree Taylor polynomial for $f$ about $x = 0$ is<br>$T_4(x) = 2 + 3x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-6}{4!}x^4$<br>$= 2 + 3x - x^2 - \frac{1}{4}x^4$ | Two terms of polynomial | **1 point** |
| | Remaining terms | **1 point** |

**Scoring notes:**

- The first point is earned for a correct fourth derivative or for $f^{(4)}(x) = -2 \cdot f'\left(x^2\right) + (-2x) f''\left(x^2\right)$.
- The second point is earned only for a completely correct expression for $f^{(4)}(x)$.
- A response that earns the first point but not the second may evaluate the presented expression for $f^{(4)}(x)$ at $x = 0$ and use the consistent nonzero value in computing the coefficient of $x^4$ in the fourth-degree Taylor polynomial.
- A polynomial that includes a nonzero third-degree term, any terms of degree greater than four, or $+\ldots$ does not earn the fourth point.

**Total for part (a) — 4 points**

**(b)** The fourth-degree Taylor polynomial for $f$ about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ of the exact value of $f(0.1)$.

| | | |
|---|---|---|
| By the Lagrange error bound, $$\left\|T_4(0.1) - f(0.1)\right\| \le \frac{\max\limits_{0\le x\le 0.1}\left\|f^{(5)}(x)\right\|}{5!}\cdot(0.1)^5 \le \frac{15}{120}\cdot\frac{1}{10^5} \le \frac{1}{10^5}$$ | Form of error bound | **1 point** |
| | Shows $\|\text{Error}\| \le \frac{1}{10^5}$ | **1 point** |

**Scoring notes:**

- The first point is earned for $\frac{\max\limits_{0\le x\le 0.1}\left|f^{(5)}(x)\right|}{5!}\cdot(0.1)^5$ or $\frac{15}{5!}(0.1)^5$. Subsequent errors in simplification will not earn the second point.
- To earn the second point a response must communicate the inequality $\text{Error} \le \frac{15}{5!}\cdot(0.1)^5 \le \frac{1}{10^5}$.
- A response that states $\text{Error} = \frac{15}{5!}\cdot(0.1)^5$ or $\text{Error} = \frac{1}{10^5}$ does not earn the second point.

**Total for part (b) 2 points**

**(c)** Let $g$ be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for $g$ about $x = 0$.

| | | |
|---|---|---|
| $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$ | $g''(x)$ | **1 point** |
| $g'(0) = e^0 \cdot f(0) = 2$<br>$g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0) = 2 + 3 = 5$<br><br>The second-degree Taylor polynomial for $g$ about $x = 0$ is<br>$T_2(x) = 4 + 2x + \frac{5}{2}x^2$. | First two terms of polynomial | **1 point** |
| | Taylor polynomial | **1 point** |

**Scoring notes:**

- The first point is earned for $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$, $g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0)$, or $g''(0) = f(0) + f'(0)$.
- A presented polynomial of the form $4 + 2x + ax^2$ earns the second point with or without any supporting work for the first two terms.
- A response that earned neither the first nor the second point only earns the third point for a polynomial of the form $a + bx + \frac{c}{2}x^2$, where $c \ne 0$ is declared to be $g''(0)$.
- A presented polynomial with no support for the coefficient of $x^2$ does not earn the third point.

- A polynomial that includes any terms of degree greater than two, or $+\ldots,$ does not earn the third point.
- Alternate solution:

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

$$e^x f(x) = \left(1 + x + \frac{x^2}{2} + \cdots\right)\left(2 + 3x - x^2 + \cdots\right) = 2 + 5x + \cdots$$

$$g(x) = \int e^x f(x)\,dx = C + 2x + \frac{5}{2}x^2 + \cdots$$

$$g(0) = 4 \Rightarrow C = 4$$

$$g(x) \approx 4 + 2x + \frac{5}{2}x^2$$

  - A response that is using this alternate solution method earns the first point for $e^x f(x) = 2 + 5x + \cdots,$ the second point for any two correct terms in a second-degree polynomial, and the third point for a completely correct second-degree Taylor polynomial with supporting work.
  - Note: There is not enough information to conclude that $f(x)$ is equal to its Maclaurin series on its interval of convergence. The second and third lines of the alternate solution are being accepted as identifications of the Maclaurin series for $e^x f(x)$ and $g(x),$ respectively.

| | |
|---|---|
| **Total for part (c)** | **3 points** |
| **Total for question 6** | **9 points** |

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$$f^{(4)}(x) = \frac{d}{dx} f'''(x) = \frac{d}{dx}\left(-2x \cdot f'(x^2)\right)$$

$$= -2f'(x^2) - 4x^2 \cdot f''(x^2)$$

$$f''(0) = -2; \quad f'''(0) = 0; \quad f^{(4)}(0) = -6$$

$$f(x) \approx 2 + 3x - x^2 - \frac{x^4}{4}$$

Response for question 6(b)

$$|E| \leq \left| \frac{15 \cdot (0.1)^5}{5!} \right| = \frac{1}{8 \cdot 10^5} < \frac{1}{10^5}$$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

**Answer QUESTION 6 part (c) on this page.**

Response for question 6(c)

$e^x \approx 1 + x$ (first 2 Taylor terms)

$f(x) \approx 2 + 3x$

$e^x f(x) \approx 2 + 5x$ (first 2 terms)

$g(x) = \int e^x f(x) \approx 2x + \frac{5x^2}{2} + C$

$g(0) = 4, \quad g(x) \approx 4 + 2x + \frac{5x^2}{2}$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$f^4(x) = \frac{d}{dx}\left(-2x f'(x^2)\right)$

$(-2x)(2x) f''(x^2)$

$f^4(x) = -4x^2 f''(x^2)$

$\frac{2(x)^0}{0!} + \frac{3(x)}{1!} + \frac{-f(0)x^2}{2!} - \frac{2x f'(0)x^3}{3!} - \frac{4x^2 f''(x^2)x^4}{4!}$

$2 + 3x - \frac{f(0)x^2}{2} - \frac{2x^4 f'(0)}{6} - \frac{4x^6 f''(0)}{4!} = P_4(x)$

Response for question 6(b)

$\frac{f^5(x)(x-c)^5}{5!} < \frac{1}{10^5}$

$\frac{15(0-.1)^5}{5!}$

$15\left(\frac{-1}{10}\right)^5$

$\frac{15\left(\frac{-1}{10}\right)^5}{5!}$

$\frac{-\cancel{15}}{10^5} \times \frac{1}{\cancel{5}\times 4\times \cancel{3}\times 2\times 1}$

$\frac{-1}{8\cdot 10^5}$ $\left(\frac{5}{10}\right)^5$ $\frac{15\left(\frac{5^5}{10^5}\right)}{5!}$

$\left(\frac{-1}{10}\right)^5$ $\boxed{8\cdot\frac{-1}{10^5} < \frac{1}{10^5}}$ ✓



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0018897

Q5178/14

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 part (c) on this page.

Response for question 6(c)

$g(\cancel{0}) = 4$

$g'(0) = e^0 f(0) = f(0)$

$g''(0) = f'(0) + f(0)$

$e^x f(x)$

$e^x f'(x) + e^x f(x)$

$\cancel{e^0} f'(0) + \cancel{e^0} f(0)$

$$\boxed{T_4(0) = 4 + f(0)x + \frac{(f'(0) + f(0))x^2}{2}}$$



Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

**Answer QUESTION 6 parts (a) and (b) on this page.**

Response for question 6(a)

Taylor polynomial $= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!}$

@ $x = 0 \rightarrow$ Maclaurin

~~f(0) = 2~~

~~f'(0) = 3~~

~~$f^{(4)}(x) =$~~

$$f^{(4)}(x) = \left(-2 \cdot f'(x^2)\right) + \left(-2x\left(\cdot f''(x^2) \cdot 2x\right)\right)$$

Response for question 6(b)

Lagrange Error bound $\leq \left| \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!} \right|$

$$\leq \left| \frac{0.5(0 - 0.1)}{5!} \right| = \frac{-0.5}{5!} \leq \frac{1}{10^5}$$

✓

Answer QUESTION 6 part (c) on this page.

Response for question 6(c)

2nd degree Taylor polynomial for g about $\underline{x=0}$ →

$$\cancel{f(0) + f'(0)x + \frac{f''(0)x^2}{2}}$$

$$\cancel{= 4 + e^0 f(0)\cdot 0 + (e^0 \cdot f(0) + e^0 \cdot f'(0))}$$

$$\cancel{= 4 +}$$

$$\boxed{= g(0) + g'(0)x + \frac{g''(0)x^2}{2}}$$

## Question 6

**Note:** Student samples are quoted verbatim and may contain spelling and grammatical errors.

### Overview

In this problem students were told that the function $f$ has derivatives of all orders for all real numbers and that $f(0) = 2,\ f'(0) = 3,\ f''(x) = -f\left(x^2\right),$ and $f'''(x) = -2x \cdot f'\left(x^2\right).$

In part (a) students were asked to find $f^{(4)}(x)$ and then to write the fourth-degree Taylor polynomial for $f$ about $x = 0.$ A correct response will use the product and chain rules to find $f^{(4)}(x) = -2 \cdot f'\left(x^2\right) + (-2x) f''\left(x^2\right) \cdot 2x.$ The response will then evaluate the first four derivatives of $f$ at $x = 0$ and use these values to write the fourth-degree Taylor polynomial $T_4(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \frac{f^{(4)}(0)}{4!} \cdot x^4,$ which is $T_4(x) = 2 + 3x - x^2 - \frac{1}{4}x^4.$

In part (b) students were asked to use the Lagrange error bound to show that the approximation of $f(0.1)$ found using the fourth-degree Taylor polynomial is within $\frac{1}{10^5}$ of the exact value, given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5.$ A correct response will indicate that the Lagrange error bound limits the absolute value of the difference between the approximation and the exact value to $\frac{\max\limits_{0 \le x \le 0.1}\left|f^{(5)}(x)\right|}{5!} \cdot (0.1)^5 \le \frac{15}{120} \cdot \frac{1}{10^5}$ which is less than $\frac{1}{10^5}.$

In part (c) students were told that $g$ is a function with $g(0) = 4$ and $g'(x) = e^x f(x)$ and asked to write the second-degree Taylor polynomial for $g$ about $x = 0.$ A correct response will use the product rule to find $g''(0) = e^0 \cdot f'(0) + e^0 \cdot f(0) = 5,$ evaluate $g'(0) = e^0 f(0) = 2,$ and then put these two values together with the given value of $g(0) = 4$ to write the polynomial $T_2(x) = 4 + 2x + \frac{5}{2!}x^2.$

**Sample: 6A**
**Score: 9**

The response earned 9 points: 4 points in part (a), 2 points in part (b), and 3 points in part (c).

In part (a) the response earned the first and second points in the second line of work. The polynomial presented in the fourth line of work earned the response the third and fourth points.

In part (b) the response earned the first and second points with the inequality presented. Note that "$E$" does not require absolute value because the term error may be used to represent the magnitude of difference.

In part (c) the response earned the first point for the alternate solution with the expression in the third line of work. The response earned the second and third points with the correct polynomial in the last line of work. The response did not lose a point for not including the $dx$ in the integral expression in the fourth line of work.

## Question 6 (continued)

**Sample: 6B**
**Score: 4**

The response earned 4 points: 1 point in part (a), no points in part (b), and 3 points in part (c).

In part (a) the response did not earn the first or second point because there is no evidence of the product rule. The response earned the third point with the first two terms of the Taylor polynomial presented in the fourth line. The response did not earn the fourth point because there are only two correct terms in the polynomial presented.

In part (b) the response did not earn the first point because the expression in the first line of work is negative. If the base of the power was positive, the response would have earned the first point. The response is not eligible to earn the second point.

In part (c) the response earned the first point with $g''(0) = f'(0) + f(0)$ in the third line. The second and third points were earned with the correct polynomial presented because the prompt states that $f(0) = 2$ and $f'(0) = 3$.

**Sample: 6C**
**Score: 2**

The response earned 2 points: 2 points in part (a), no points in part (b), and no points in part (c).

In part (a) the response earned the first and second points with the derivative presented in the last line of work. The response did not earn the third or fourth points because the coefficients in the Taylor polynomial are not evaluated.

In part (b) the response did not earn the first point because the Lagrange error bound is not properly used. The response is ineligible to earn the second point.

In part (c) the response did not earn the first point because $g''(x)$ is not presented. The response did not earn the second or third points because the coefficients in the Taylor polynomial are not presented.